SELF-ADJOINT OPERATORS AND NONTRIVIAL ZEROS OF DIRICHLET L-FUNCTION

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Abstract. We give a kind of self-adjoint operator, whose spectrums are the set $S_\chi = \{i(\rho - \frac{1}{2}) \mid \rho$ is nontrivial zeros of L-function $L(\chi, s)\}$.

Alain Connes[1] first proved the critical zeros of Hecke L-function corresponds to the the spectrums of a suitable operator. His main methods is to deal with the relation between the L^2 of adele class space and the L^2 of idele class space by the adelic Riemann-Roth theorem. In [2, Thm.4.16], Connes, Consani and Marcolli give the spectral realization of zeros of Dirichlet L-function as the action of $\mathbb{R}_+^*$ on a suitable space.

Motivated by Alain Connes's spectral interpretation for the zeros of L-functions, Ralf Meyer[6] developed an alternative spectral interpretation for the poles and zeros and for André Weil's explicit formula, but no longer directly related to the Riemann hypothesis. André Weil's explicit formula also has relation with Riesz potential(see [4]). In [7, Corollary4.2], R. Meyer proved that the eigenvalues of the transpose D_-^t of the operator D_-(induced by D on some function space) acting on some space of continuous linear functionals are exactly the nontrivial zeros of $\zeta(s)$. Furthermore, Xian-Jin Li [5] proved that every nontrivial zero of the zeta function is indeed an eigenvalue of D_-. His method has been generalized to Dirichlet L function by Dongsheng Wu[9]. Liming Ge, Xian-Jin Li, Dongsheng Wu and Boqing Xue in [3] proved that the correspondence between the set of eigenvalues of D_- acting on $\mathcal{H}$ and the set of nontrivial zeros of $\zeta(s)$ is one-to-one.

Inspired by the above results, we find a suitable self-adjoint operator which are related with the nontrivil zeros of Dirichlet L-function. The method we do here is an old idea, called Hilbert-Pólya conjecture.

1. The self-adjoint operator

Denote $\mathbb{R}_+^\times = (0, \infty)$, $C^\infty(\mathbb{R}_+^\times)$ the set of smooth complex valued functions on $\mathbb{R}_+^\times$. Let

$$\mathcal{H}_0 = \{f \in C^\infty(\mathbb{R}_+^\times) \mid \lim_{x\to\infty} x^m f^{(n)}(x) = 0 \text{ and } f^{(n)}(0) := \lim_{x\to 0^+} f^{(n)} \text{ exists}, \forall m, n \in \mathbb{N}\}.$$

$$\mathcal{H}_\cap := \{f \in \mathcal{H}_0 \mid \int_0^\infty f(x)dx = 0, f(0) = 0 \text{ and } f^{(2n+1)}(0) = 0 \text{ for } n \in \mathbb{N}\}.$$

$$\mathcal{H}_- := \{f \in \mathcal{H}_0 \mid f^{(n)}(0) = 0 \text{ for } n \in \mathbb{N}\}.$$

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Let χ be a primitive Dirichlet character. Define
$\mathcal{H}_{\cap}^{\chi} := \{f \in \mathcal{H}_0 \mid f^{(2n+1)}(0) = 0 \text{ if } \chi(-1) = 1, f^{(2n)}(0) = 0 \text{ if } \chi(-1) = -1, \forall n \in \mathbb{N}\}$.

The inner product on $\mathcal{H}_0$ is defined by

$$\langle f(x), g(x)\rangle = \int_0^\infty f(x)\overline{g(x)}dx.$$

Then $\mathcal{H}_0$ is a unitary space, i.e., a complex space with inner product. We define two self-adjoint operators $\mathcal{D}, \mathcal{M}$ on $\mathcal{H}_0$ by

$$\mathcal{D}f(x) = -if'(x), \quad \mathcal{M}f(x) = xf(x).$$

That is, for $f, g \in \mathcal{H}_0$ we have

$$\langle \mathcal{D}f, g\rangle = \langle f, \mathcal{D}g\rangle, \quad \langle \mathcal{M}f, g\rangle = \langle f, \mathcal{M}g\rangle.$$

The above equations are easy to compute, leaving it to the reader, or see [8, Example7.1.5,7.1.6]. It is easy to check that

$$\mathcal{M}\mathcal{D} - \mathcal{D}\mathcal{M} = i. \tag{1.1}$$

Our key idea is the following

Theorem 1.1. *$\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator on $\mathcal{H}_0$.*

Proof. From equation(1.1), for $f, g \in \mathcal{H}_0$ we have

$$\langle \mathcal{M}\mathcal{D}f, g\rangle = \langle f, \mathcal{M}\mathcal{D}g\rangle + i\langle f, g\rangle.$$

Then

$$\begin{aligned}\langle (\mathcal{M}\mathcal{D} - \frac{i}{2})f, g\rangle &= \langle \mathcal{M}\mathcal{D}f, g\rangle - \frac{i}{2}\langle f, g\rangle \\ &= \langle f, \mathcal{M}\mathcal{D}g\rangle + i\langle f, g\rangle - \frac{i}{2}\langle f, g\rangle \\ &= \langle f, \mathcal{M}\mathcal{D}g\rangle + \frac{i}{2}\langle f, g\rangle \\ &= \langle f, (\mathcal{M}\mathcal{D} - \frac{i}{2})g\rangle.\end{aligned}$$

Hence, $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator. □

Lemma 1.2. *$\mathcal{H}_-$ is invariant subspace of $\mathcal{D}, \mathcal{M}$, hence, $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator on it.*

Proof. It is easy to check that for $f \in \mathcal{H}_-$, we have $\mathcal{D}f, \mathcal{M}f \in \mathcal{H}_-$. Hence, $\mathcal{D}, \mathcal{M}$ are operators on $\mathcal{H}_-$. Also, $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator on it. □

For $f \in \mathcal{H}_\cap$, define the operator $\mathcal{Z}$ by

$$(\mathcal{Z}f)(x) = \sum_{n=1}^\infty f(nx),$$

and for $f \in \mathcal{H}_\cap^\chi$, define the operator $\mathcal{Z}_\chi$ by

$$(\mathcal{Z}_\chi f)(x) = \sum_{n=1}^\infty \chi(n)f(nx).$$

Then we have $\mathcal{Z}\mathcal{H}_\cap \subset \mathcal{H}_-, \mathcal{Z}_\chi \mathcal{H}_\cap^\chi \subset \mathcal{H}_-$(see [9, Thm.2.9]). Let $(\mathcal{Z}\mathcal{H}_\cap)^\perp, (\mathcal{Z}_\chi \mathcal{H}_\cap^\chi)^\perp$ be the orthogonal complement in $\mathcal{H}_-$. Then we have

$$\mathcal{H}_- = (\mathcal{Z}\mathcal{H}_\cap)^\perp \bigoplus \mathcal{Z}\mathcal{H}_\cap = (\mathcal{Z}_\chi \mathcal{H}_\cap^\chi)^\perp \bigoplus \mathcal{Z}_\chi \mathcal{H}_\cap^\chi. \tag{1.2}$$

Theorem 1.3. *Under the canonical isomorphisms*

$$(\mathcal{Z}\mathcal{H}_\cap)^\perp \simeq \mathcal{H}_-/\mathcal{Z}\mathcal{H}_\cap, \quad (\mathcal{Z}_\chi \mathcal{H}_\cap^\chi)^\perp \simeq \mathcal{H}_-/\mathcal{Z}_\chi \mathcal{H}_\cap^\chi,$$

we have that $\mathcal{H}_-/\mathcal{Z}\mathcal{H}_\cap$ and $\mathcal{H}_-/\mathcal{Z}_\chi \mathcal{H}_\cap^\chi$ are unitary spaces. Further, $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator of them.

Proof. Since $\mathcal{Z}\mathcal{H}_\cap, \mathcal{Z}_\chi \mathcal{H}_\cap^\chi$ are invariant spaces of $\mathcal{M}\mathcal{D} - \frac{i}{2}$ and $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator of $\mathcal{H}_-$, we have $(\mathcal{Z}\mathcal{H}_\cap)^\perp, (\mathcal{Z}_\chi \mathcal{H}_\cap^\chi)^\perp$ are invariant spaces of $\mathcal{M}\mathcal{D} - \frac{i}{2}$. Furthermore, $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator of them. □

Remark 1.4. $\mathcal{Z}\mathcal{H}_\cap, \mathcal{Z}_\chi \mathcal{H}_\cap^\chi$ are invariant spaces of $\mathcal{M}\mathcal{D}$, but they are not invariant spaces under $\mathcal{M}$ or $\mathcal{D}$ lonely.

2. Spectral interpretation of L-function

Theorem 2.1. *Let $S = \{i(\rho - \frac{1}{2}) \mid \rho$ is nontrivial zeros of $\zeta(s)\}$ and $S_\chi = \{i(\rho - \frac{1}{2}) \mid \rho$ is nontrivial zeros of $L(\chi, s)\}$. Then the spectrum of $\mathcal{M}\mathcal{D} - \frac{i}{2}$ on $\mathcal{H}_-/\mathcal{Z}\mathcal{H}_\cap$ is S and on $\mathcal{H}_-/\mathcal{Z}_\chi \mathcal{H}_\cap^\chi$ is S_χ.*

Proof. This is a corollary of Theorem1.2, Theorem1.3 in [9]. □

Theorem 2.2. *The Riemann hypothesis is true.*

Proof. Since $\mathcal{M}\mathcal{D} - \frac{i}{2}$ is a self-adjoint operator, we have $S, S_\chi \subset \mathbb{R}$, which implies the Riemann hypothesis. □

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